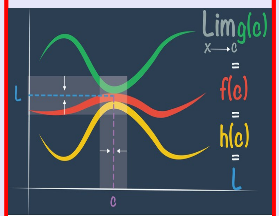
Calculus I

Lecture 12



Feb 19-8:47 AM

Given $f(x) = \frac{2x^2}{x^2-1}$, $f'(x) = \frac{-4x}{(x^2-1)^2}$, $f''(x) = \frac{4(3x^2+1)}{(x^2-1)^3}$

Graph $f(x)$, give all details as shown in the lecture.

1) Domain $x^2-1 \neq 0$ $x \neq \pm 1$

$(-\infty,-1) \cup (-1,1) \cup (1,\infty)$

2) V.A. & H.A.

$x=1$ $y=2$

$x=-1$

3) All intercepts

y-Int $(0,0)$, x-Int $(0,0)$ Twice

4) $f(-x) = \frac{2(-x)^2}{(-x)^2-1} = \frac{2x^2}{x^2-1} = f(x)$

even

Sym. y-axis

5) C.N.

$f'(x) = 0 \to x = 0$

$f'(x)$ und. $\to x = \pm 1$

6) P.I.P. location

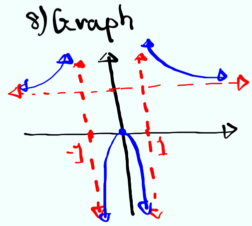
$f''(x) = 0 \to$ None

$f''(x)$ und. $\to \pm 1$

7) Sign chart

x	$-\infty$		-1		0		1		∞
$f'(x)$		$+$	$\circ$	$+$	$\bullet$	$-$	$\circ$	$-$	
$f''(x)$		$+$	$\circ$	$-$		$-$	$\circ$	$+$	
$f(x)$		$\nearrow$	$\circ$	$\nearrow$		$\searrow$	$\circ$	$\searrow$	

8) Graph



9) Range

$(-\infty, 0] \cup (2, \infty)$

Jan 26-8:06 AM

find y':

$$2x = \frac{x^2y+1}{y-1}$$

Cross-Multiply

$$2x(y-1) = x^2y+1$$

Distribute

$$2xy - 2x = x^2y + 1$$

$$\frac{d}{dx}\left[2xy - 2x\right] = \frac{d}{dx}\left[x^2y + 1\right]$$

$$2\left[1 \cdot y + x \cdot \frac{dy}{dx}\right] - 2 = 2x \cdot y + x^2\frac{dy}{dx} + 0$$

$$2y + \boxed{2x\frac{dy}{dx}} - 2 = 2xy + x^2\frac{dy}{dx}$$

$$(2x - x^2)\frac{dy}{dx} = 2xy - 2y + 2$$

$$\frac{dy}{dx} = \frac{2xy - 2y + 2}{2x - x^2}$$

Jan 26-8:25 AM

find y'':

$$2x^3 - 3y^2 = 1$$

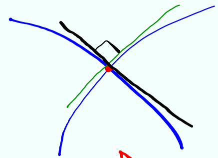
$$6x^2 - 6y\,y' = 0 \qquad y' = \frac{-6x^2}{-6y} \qquad \boxed{y' = \frac{x^2}{y}}$$

$$y'' = \frac{2x \cdot y - x^2 \cdot y'}{y^2} = \frac{2xy - x^2 \cdot \frac{x^2}{y}}{y^2} \qquad \text{LCD} = y$$

$$\boxed{y'' = \frac{2xy^2 - x^4}{y^3}}$$

Jan 26-8:31 AM

Orthogonal curves have perpendicular tangent lines at the intersection points.

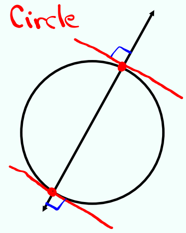


Product of slopes $= -1$ is they are perpendicular to each other.

$x^2+y^2=r^2$ Circle

$ax+by=0$ Line

$by=-ax$



$$x^2+y^2=r^2$$
$$2x+2y\frac{dy}{dx}=0$$
$$\frac{dy}{dx}=\frac{-x}{y}$$

$$ax+by=0$$
$$a+b\frac{dy}{dx}=0$$
$$\frac{dy}{dx}=\frac{-a}{b}$$

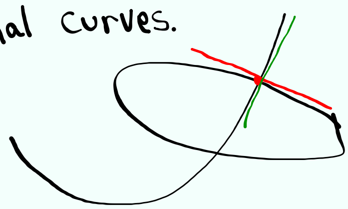
$$\frac{-x}{y}\cdot\frac{-a}{b}=\frac{ax}{by}=\frac{ax}{-ax}=-1$$

Product of slopes (Derivatives) $=-1$

Curves are orthogonal.

Jan 26-8:37 AM

Show $y=cx^2$ and $x^2+2y^2=K$ are orthogonal curves.



$$y=cx^2$$
$$\boxed{y'=2cx}$$

$$x^2+2y^2=K$$
$$2x+4yy'=0$$
$$\boxed{y'=\frac{-x}{2y}}$$

Show product of first derivatives is equal to -1.

$$\cancel{2}cx\cdot\frac{-x}{\cancel{2}y}=-1 \checkmark$$

$$\frac{-cx^2}{y}=-\frac{y}{y}=\boxed{-1}$$

Curves are orthogonal to each other.

Jan 26-8:47 AM

Use Quadratic Approximation to estimate $\sqrt{16.01}$.

From Calculator
$\sqrt{16.01} \approx 4.001249...$

From Common Sense
$\sqrt{16.01} \approx \sqrt{16} = 4$

$f(x) = \sqrt{x}$

$a = 16$

$f(16) = \sqrt{16} = 4$

$f'(x) = \dfrac{1}{2\sqrt{x}}$

$f'(16) = \dfrac{1}{2\sqrt{16}} = \dfrac{1}{8}$

$f''(x) = \dfrac{-1}{4x\sqrt{x}}$

$f''(16) = \dfrac{-1}{4(16)\sqrt{16}} = \dfrac{-1}{256}$

$$f(x) \approx f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2$$

$$\sqrt{x} \approx f(16) + f'(16)(x-16) + \frac{f''(16)}{2}(x-16)^2$$

$$\sqrt{x} \approx 4 + \frac{1}{8}(x-16) - \frac{1}{512}(x-16)^2$$

$$\sqrt{16.01} \approx 4 + \frac{1}{8}(16.01-16) - \frac{1}{512}(16.01-16)^2$$

$$= 4 + \frac{1}{8}(.01) - \frac{1}{512}(.01)^2$$

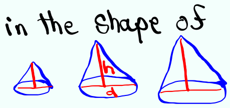
$$= \frac{25612}{6401}$$

$$= \boxed{4.001}249805$$

Jan 26-8:55 AM

Gravel is being dumped at the rate of 30 ft^3/m. $\dfrac{dV}{dt} = 30\,ft^3/m$. $V = \dfrac{\pi r^2 h}{3}$

It is forming a pile in the shape of a right-circular cone.



the diameter and height are always equal.
$2r = h$
$r = \dfrac{h}{2}$

How fast is the height increasing $\dfrac{dh}{dt}$?
when the pile is 10 ft high? $\boxed{h=10}$

$V = \dfrac{\pi r^2 h}{3}$ $\qquad V = \dfrac{\pi\left(\frac{h}{2}\right)^2 h}{3}$

$V = \dfrac{\pi h^3}{12}$

$\dfrac{dV}{dt} = \dfrac{\pi}{\cancel{12}} \cdot \cancel{3}h^2\, \dfrac{dh}{dt}$

$30 = \dfrac{\pi}{4} \cdot 10^2 \cdot \dfrac{dh}{dt}$

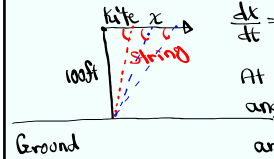
$120 = 100\pi \dfrac{dh}{dt}$

$\dfrac{\overset{6}{\cancel{120}}}{\underset{5}{\cancel{100}}\pi} = \dfrac{dh}{dt}$

$$\boxed{\frac{dh}{dt} = \frac{6}{5\pi}\ ft/min.}$$

Jan 26-9:07 AM

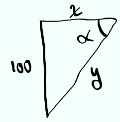
A Kite 100 ft above the ground moves horizontally at the speed of 8 ft/s.

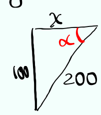


$\frac{dx}{dt} = 8$ ft/s.

At what rate the angle between the string and the horizontal decreasing when 200 ft of string has been let out?

$\frac{d\alpha}{dt}$ when $y = 200$





$\tan\alpha = \frac{100}{x} \rightarrow 100x^{-1}$

$\sec^2\alpha \cdot \frac{d\alpha}{dt} = 100\cdot(-1)x^{-2}\cdot\frac{dx}{dt}$

$\sec^2\alpha \, \frac{d\alpha}{dt} = \frac{-100}{x^2}\,\frac{dx}{dt}$

$\frac{4}{3}\,\frac{d\alpha}{dt} = \frac{-100}{30000}\cdot 8$ (100, 2)

$100^2 + x^2 = 200^2$

$x^2 = 200^2 - 100^2$

$x^2 = 30000$

$\sec\alpha = \frac{200}{x}$

$\sec^2\alpha = \frac{200^2}{x^2}$

$= \frac{200\cdot 200}{30000}$

$= 4/3$

$$\boxed{\frac{d\alpha}{dt} = \frac{-1}{50} \text{ Rad./S.}}$$

Jan 26-9:19 AM

Find two numbers whose difference is 100

x & y $\qquad x - y = 100$

and whose product is a minimum.

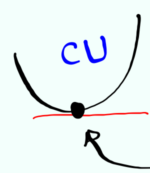
xy $\qquad y = x - 100$

$f(x) = x(x - 100)$

$= x^2 - 100x$

$f'(x) = 2x - 100$

$f''(x) = 2 > 0$



$2x - 100 = 0$

$x = 50$

$y = x - 100$

$= 50 - 100$

$\boxed{y = -50}$

Two numbers are 50 & −50.

optimization.

Jan 26-9:34 AM

Find $\boxed{\text{two positive numbers}}$ whose $\boxed{\text{product is } 100}$

$x \,\&\, y$ $\quad x>0,\ y>0$ $\quad xy = 100$

and $\boxed{\text{whose sum}}$ is minimum.

$y = \dfrac{100}{x}$

$x + y = x + \dfrac{100}{x}$

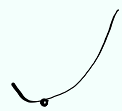
$y = \dfrac{100}{10}$

$\boxed{y = 10}$

$f(x) = x + \dfrac{100}{x}$ $\qquad f(x) = x + 100x^{-1}$

$f'(x) = 1 - 100x^{-2}$

$f''(x) = 200x^{-3} = \dfrac{200}{x^3} > 0$ C.U.



Min

$f'(x) = 0$ or undef.

$1 - 100x^{-2} = 0$

$1 - \dfrac{100}{x^2} = 0$

$\dfrac{x^2 - 100}{x^2} = 0$

$x^2 - 100 = 0$

$x = \pm 10$ $\quad \boxed{x = 10}$

Two numbers are 10 & 10

$f'(x) = 1 - \dfrac{100}{x^2} = \dfrac{x^2 - 100}{x^2}$

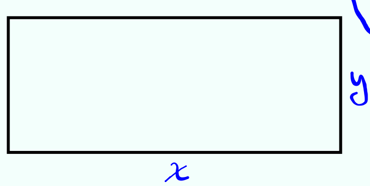
$x = 9$ $\quad x = 11$

$\searrow$ 10 $\nearrow$

Minimum

Jan 26-10:06 AM

Find dimensions of a rectangular shape with $\boxed{\text{perimeter } 100 \text{ m}}$ and $\boxed{\text{area}}$ as large as possible.



$2x + 2y = 100$

$x + y = 50$

$y = 50 - x$

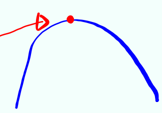
xy

$x(50 - x)$

$f(x) = x(50 - x)$ $\qquad f(x) = 50x - x^2$

$f'(x) = 50 - 2x$

$f''(x) = -2 < 0$ CD



Max

$f'(x) = 0$ $\qquad 50 - 2x = 0$

$\boxed{x = 25}$

$y = 50 - 25$

$\boxed{y = 25}$

25 ft by 25 ft

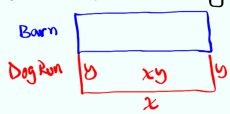
Jan 26-10:16 AM

A farmer has $\boxed{500 \text{ ft}}$ of fencing.

He wants to make a rectangular dog run next to the wall of a barn.

Find the dimensions that give the largest area for the dog.

No fence against the wall needed.



$x + 2y = 500$

$xy = (500 - 2y)y$

$f(y) = (500 - 2y)y$

$f(y) = 500y - 2y^2$

$f'(y) = 500 - 4y$

$f''(y) = -4 < 0$ CD

Max

$f'(y) = 0$

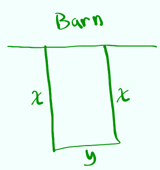
$500 - 4y = 0$

$\boxed{y = 125}$

$x = 500 - 2(125)$

$\boxed{x = 250}$

125 ft by 250 ft



$2x + y = 500$

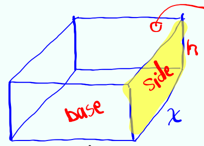
$y = 500 - 2x$

$f(x) = xy = x(500 - 2x)$

Jan 26-10:22 AM

You have $\boxed{1200 \text{ cm}^2}$ of matrials.

You wish to make a box with square base and open top.



$V = LWH$

$= x^2 h$

$x > 0$

Find dimensions of the box with largest volume

Base + Sides $= 1200$

$x^2 + 4xh = 1200$

$h = \dfrac{1200 - 20^2}{4(20)} = \boxed{10}$

$4xh = 1200 - x^2$

$h = \dfrac{1200 - x^2}{4x}$

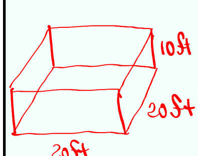
$V = x^2 \cdot \dfrac{1200 - x^2}{4x}$

$V(x) = \dfrac{x(1200 - x^2)}{4}$

$V(x) = \dfrac{1}{4}\left[1200x - x^3\right]$

$V'(x) = \dfrac{1}{4}\left[1200 - 3x^2\right]$

$V''(x) = \dfrac{1}{4}\left[-6x\right] < 0$



$V'(x) = 0$

Max

CD

$\dfrac{1}{4}\left[1200 - 3x^2\right] = 0$

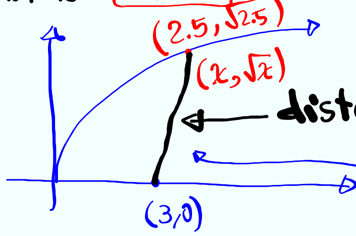
$\vdots$

$x^2 = 400$

$\boxed{x = 20}$

Jan 26-10:35 AM

Find a point on the graph of $y=\sqrt{x}$ that is $\boxed{\text{closest}}$ to the point $(3,0)$.



distance has to be minimum.

$d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$

$d=\sqrt{(x-3)^2+(\sqrt{x}-0)^2}$

$=\sqrt{\underbrace{(x-3)^2+x}_{\text{Minimum}}}$

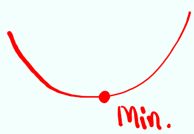
$f(x)=(x-3)^2+x$

$f'(x)=2(x-3)^1\cdot 1+1$

$=2(x-3)+1$

$=2x-5$

$f''(x)=2>0$ CU



$f'(x)=0$

$2x-5=0$

$x=5/2$

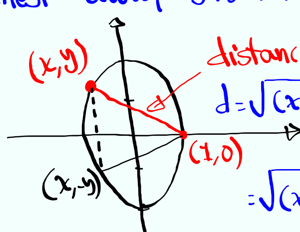
Jan 26-10:48 AM

Find $\boxed{\text{the points}}$ on the graph of $4x^2+y^2=4$ that are farthest away from the point $(1,0)$.

$4x^2+y^2=4$

Ellipse

x	y
0	± 2
± 1	0



distance to be max.

$d=\sqrt{(x-1)^2+(y-0)^2}$

$=\sqrt{(x-1)^2+y^2}$

$=\sqrt{\underbrace{(x-1)^2+4-4x^2}_{\text{Maximize}}}$

$f(x)=(x-1)^2+4-4x^2$

$f'(x)=2(x-1)-8x=-6x-2$

$f''(x)=-6<0$

$f'(x)=0$

$-6x-2=0$

$\boxed{x=-\frac{1}{3}}$

$4\left(-\frac{1}{3}\right)^2+y^2=4$

$\frac{4}{9}+y^2=4$

$y^2=4-\frac{4}{9}=\frac{32}{9}$

$\boxed{y=\pm\frac{4\sqrt{2}}{3}}$

$\boxed{\left(-\frac{1}{3},\pm\frac{4\sqrt{2}}{3}\right)}$

Jan 26-10:58 AM

Doing Reverse:

Find $f(x)$ if $f'(x) = 2x + 6$

$$\boxed{f(x) = x^2 + 6x + C}$$

Find $f(x)$ if $f'(x) = 3x^2 - 2x + 8$ and $\boxed{f(1) = 10}$

$f(x) = x^3 - x^2 + 8x + C$

$f(1) = \cancel{1^3} - \cancel{1^2} + 8(1) + C = 10$

$8 + C = 10 \rightarrow \boxed{C = 2}$

$$\boxed{f(x) = x^3 - x^2 + 8x + 2}$$

Jan 26-11:10 AM

Find $f(x)$ if $f(\pi/3) = 4$ and

$f'(x) = 2\cos x + \sec^2 x$

over $(-\frac{\pi}{2}, \frac{\pi}{2})$.

$f(x) = 2 \cdot \sin x + \tan x + C$

$f(\frac{\pi}{3}) = 2 \cdot \sin \frac{\pi}{3} + \tan \frac{\pi}{3} + C = 4$ ($\frac{\pi}{3} \to 60^\circ$, $\frac{\pi}{3} \to 60$)

$\cancel{2} \cdot \frac{\sqrt{3}}{\cancel{2}} + \sqrt{3} + C = 4$

$2\sqrt{3} + C = 4$ $\quad \boxed{C = 4 - 2\sqrt{3}}$

$$\boxed{f(x) = 2\sin x + \tan x + 4 - 2\sqrt{3}}$$

Jan 26-11:16 AM

Find $f(x)$ if $f(0)=3$, $\boxed{f'(0)=4}$, and

$f''(x) = \sin x + \cos x$

$f'(x) = -\cos x + \sin x + C$

$f'(0) = -\cos 0 + \sin 0 + C = 4$

$-1 + C = 4 \rightarrow \boxed{C=5}$

$f'(x) = -\cos x + \sin x + 5$

$f(x) = -\sin x - \cos x + 5x + C$

$f(0) = -\sin 0 - \cos 0 + 5(0) + C = 3$

$-1 + C = 3$

$\boxed{C=4}$

$$\boxed{f(x) = -\sin x - \cos x + 5x + 4}$$

Jan 26-11:22 AM

$f''(x) = 2 + \cos x$

$f(0) = -1$, $f(\frac{\pi}{2}) = 0$

Find $f(x)$.

$f'(x) = 2x + \sin x + C$

$f(x) = x^2 - \cos x + Cx + D$

$f(0) = 0^2 - \cos 0 + C(0) + D = -1$

$-1 + D = -1$ $\boxed{D=0}$

$f(x) = x^2 - \cos x + Cx$

$f(\frac{\pi}{2}) = (\frac{\pi}{2})^2 - \cos\frac{\pi}{2} + C(\frac{\pi}{2}) = 0$

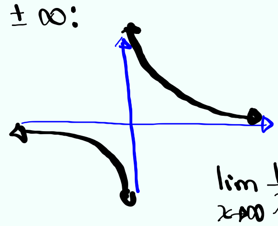
$\frac{\pi^2}{4} + C\frac{\pi}{2} = 0$ $\quad C = \frac{-\frac{\pi^2}{4}}{\pi/2} = -\frac{\pi}{2}$

$$\boxed{f(x) = x^2 - \cos x - \frac{\pi}{2}x}$$

Jan 26-11:29 AM

limits at $\pm\infty$:

$f(x) = \frac{1}{x}$



$$\lim_{x\to\infty} \frac{1}{x} = 0$$

$$\lim_{x\to-\infty} \frac{1}{x} = 0$$

Find $\lim_{x\to\infty} \frac{2x+1}{3x-5} = \frac{2(\infty)+1}{3(\infty)-5} = \frac{\infty}{\infty}$ I.F.

$$\lim_{x\to\infty} \frac{2x+1}{3x-5} = \lim_{x\to\infty} \frac{\frac{2x+1}{x}}{\frac{3x-5}{x}} = \lim_{x\to\infty} \frac{2+\frac{1}{x}}{3-\frac{5}{x}} = \frac{2+0}{3-0} = \boxed{\frac{2}{3}}$$

Divide everything by x to the highest power.

Let $x = 1000$

$\frac{2(1000)+1}{3(1000)-5} \approx .668$

Let $x = 1000000$

$\frac{2(1000000)+1}{3(1000000)-5} \approx .667$

$\approx .667$

Jan 26-11:36 AM

Evaluate $\lim_{x\to\infty} \frac{x^2-3x+5}{4x^2-25} = \frac{\infty^2-3(\infty)+5}{4(\infty)^2-25} = \frac{\infty}{\infty}$

Divide by x^2, and Simplify

$$= \lim_{x\to\infty} \frac{\frac{x^2-3x+5}{x^2}}{\frac{4x^2-25}{x^2}} = \lim_{x\to\infty} \frac{1-\frac{3}{x}+\frac{5}{x^2}}{4-\frac{25}{x^2}}$$

$$= \boxed{\frac{1}{4}}$$

Jan 26-11:45 AM

Evaluate $\lim_{x\to\infty} \frac{x}{\sqrt{4x^2+1}}$ $\approx \lim_{x\to\infty} \frac{x}{\sqrt{4x^2}}$

Let $x=1000$

$\frac{1000}{\sqrt{4(1000)^2+1}} = .4999999375$

$= \lim_{x\to\infty} \frac{x}{2x}$

$= \lim_{x\to\infty} \frac{1}{2} = \boxed{\frac{1}{2}}$

$$\lim_{x\to\infty} \frac{x}{\sqrt{4x^2+1}} = \lim_{x\to\infty} \frac{\frac{x}{x}}{\frac{\sqrt{4x^2+1}}{x}} = \lim_{x\to\infty} \frac{1}{\sqrt{\frac{4x^2+1}{x^2}}}$$

Divide by x

$$= \lim_{x\to\infty} \frac{1}{\sqrt{4+\cancel{\frac{1}{x^2}}^{\,0}}} = \frac{1}{\sqrt{4}} = \frac{1}{2}$$

$\sqrt{A^2+B^2} \neq \sqrt{A^2}+\sqrt{B^2}$

Jan 26-11:49 AM

Evaluate $\lim_{x\to\infty} \left(\sqrt{x^2+4x} - x\right) = \infty - \infty$

I.F.

$\frac{0}{0}, \frac{\infty}{\infty}, \infty - \infty$

Use conjugate to rationalize.

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